

PROJECTIVE BUNDLES OVER SMOOTH COMPACT TORIC SURFACES

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ABSTRACT. Let E be the Whitney sum of complex line bundles over a smooth compact toric surface S . Then, the projectivization $P(E)$ is a toric manifold, and it is called a *projective bundle* over S . If the cohomology ring of a quasitoric manifold M is isomorphic to that of a projective bundle $P(E)$, then the orbit space of M can be identified with that of $P(E)$; furthermore, M is equivalent or homeomorphic to some projective bundle $P(E')$. Moreover, such quasitoric manifolds are classified by their cohomology rings up to homeomorphism in some cases.

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1. INTRODUCTION

A *toric variety* is a normal algebraic variety of complex dimension n with an action of the algebraic torus $(\mathbb{C}^*)^n$ having an open dense orbit. A compact smooth toric variety is called a *toric manifold*. A typical example of a toric manifold is the complex projective space $\mathbb{C}P^n$ with a linear action of $(\mathbb{C}^*)^n$. A toric variety of complex dimension 2 is called a *toric surface*. It is well known that every smooth compact toric surface is projective and that it can be obtained by blow-ups from either $\mathbb{C}P^2$ or one of the Hirzebruch surfaces as an algebraic variety.

Let B be a toric manifold of complex dimension k . Let $L_i \rightarrow B$ be a complex line bundle over B for $i = 0, \dots, n$. Note that each L_i has a \mathbb{C}^* -action as a scalar multiplication, and hence, the Whitney sum $E = \bigoplus_{i=0}^n L_i$ has a $(\mathbb{C}^*)^{k+n+1}$ -action. Hence, the projectivization $P(E)$ of E has

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an induced $(\mathbb{C}^*)^{k+n}$ -action and is also a toric manifold. The space $P(E)$ is called a projective bundle over B . From this viewpoint, we can construct an interesting class of toric manifolds as follows. Starting with B as a complex projective space and repeating the above construction $h - 1$ times ($h \in \mathbb{N}$), we obtain a new toric manifold, which is called an *h -stage generalized Bott manifold* (see [8] for details). We note that a generalized Bott manifold is the total space of iterated projective space bundles over a projective space. In particular, for an iterated projective space bundle over $\mathbb{C}P^1$, when the complex dimension of fiber is equal to 1 at each fibration, the total space is simply referred to as a *Bott manifold*.

We observe that $\mathbb{C}P^2$ and any two-stage Bott manifold are smooth compact toric surfaces. Hence, two-stage generalized Bott manifolds over $\mathbb{C}P^2$ and three-stage Bott manifolds can be regarded as projective bundles over smooth compact toric surfaces. Therefore, we study a class of projective bundles over general toric surfaces.

On the other hand, if a toric manifold is projective, then the algebraic torus action of $(\mathbb{C}^*)^n$ induces a locally standard action of the compact subtorus $T^n = (S^1)^n$. Moreover, the orbit space X/T^n can be identified with a simple polytope¹. The identification means that there is an orbit map $\rho: X \rightarrow P$ that maps every k -dimensional orbit to a point in the interior of a codimension- k face of P for $k = 0, \dots, n$.

The topological analogue of a projective toric manifold, which is now called a *quasitoric manifold*, was introduced by Davis and Januszkiewicz [13]². A $2n$ -dimensional closed smooth manifold M is called a quasitoric manifold if it has a locally standard action of a compact torus $T^n = (S^1)^n$ of dimension n , and the orbit space M/T^n can be identified with a simple polytope of dimension n . Unlike projective toric manifolds, quasitoric manifolds do not necessarily admit almost complex structures. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not a toric manifold, while it is a quasitoric manifold with an appropriate T^2 -action. Hence, the class of quasitoric manifolds is larger than that of projective toric manifolds.

In this study, we investigate a projective bundle over a smooth compact toric surface S as a quasitoric manifold, as well as the necessary condition when a quasitoric manifold becomes a projective bundle over S . We shall prove that the cohomology ring of a quasitoric manifold can determine whether it becomes a projective bundle over S .

Theorem 1.1. *Let $P(E)$ be a projective bundle over a smooth compact toric surface S with fiber $\mathbb{C}P^n$. Let M be a quasitoric manifold whose cohomology ring is isomorphic to $H^*(P(E))$. Then, M is*

¹More precisely, the orbit space X/T^n as a simple polytope is so realized in \mathbb{R}^n that all vertices are in the integer lattice of \mathbb{R}^n and all inward vectors of facets form an integral basis for \mathbb{Z}^n at each vertex.

²The authors would like to indicate that the notion of quasitoric manifolds originally appeared under the name “toric manifolds” in [13]. Later, it was renamed in [3] in order to avoid confusion with smooth compact toric varieties. As far as the authors know, there has been a dispute about the terminology. The authors have no preference; however, in this paper, they follow the terminology used in their previous papers.

- (1) *homeomorphic to a two-stage generalized Bott manifold when $n = b_2(S) = 1$, or*
- (2) *equivalent to some projective bundle $P(E')$ over S' , where S' is a toric surface diffeomorphic to S when $n \geq 2$ or $b_2(S) \geq 2$.*

In particular, if M is a toric manifold, then M is equivalent to a projective bundle over a smooth compact toric surface.

We are also interested in classifying such manifolds topologically or smoothly. The smooth classification of generalized Bott manifolds has been considered by the first author and his collaborators in several papers. We refer the reader to a survey paper [7] on this topic. Remarkably, the results lead us to conjecture that all toric manifolds are smoothly classified by their cohomology rings. This problem is now called the *cohomological rigidity problem* for toric manifolds.

In order to solve the problem, the next natural step is to ask whether projective bundles over smooth compact toric surfaces that are not complex projective spaces are classified by their cohomology rings. We obtain the following answers:

Theorem 1.2. *Let E_i be the Whitney sum of complex line bundles over a smooth compact toric surface S_i for $i = 1, 2$. Assume that the projective bundles $P(E_1)$ and $P(E_2)$ have the same cohomology rings. Then, $P(E_1)$ and $P(E_2)$ are diffeomorphic provided that*

- (1) $\dim_{\mathbb{R}}(P(E_1)) = \dim_{\mathbb{R}}(P(E_2)) = 6$, or
- (2) $b_2(S_1) = b_2(S_2) \leq 10$.

In addition, the topological classification of quasitoric manifolds is an interesting problem. Motivated by the cohomological rigidity problem for toric manifolds, one may ask whether quasitoric manifolds are classified by their cohomology rings. A few results have been obtained for this problem: all real 4-dimensional quasitoric manifolds and all quasitoric manifolds with second Betti number 2 are classified topologically, that is, up to homeomorphism, by their cohomology rings (see [11]). In general, the cohomological rigidity problem for quasitoric manifolds is more complex than that for toric manifolds. Nevertheless, if the combinatorial type of the orbit space is determined by the cohomology ring of a quasitoric manifold, the cohomological rigidity problem for such a quasitoric manifold can be addressed. In this paper, we will show that if the cohomology ring of a quasitoric manifold M is isomorphic to that of a projective bundle $P(E)$ over a smooth compact toric surface S , then M also admits an equivariant (topological) bundle structure, and its orbit space M/T^n is always combinatorially equivalent to the product of a simplex and a polygon, which is the orbit polytope of $P(E)$. Furthermore, we have the following theorem.

Theorem 1.3. *Let $P(E)$ be a projective bundle over a 4-dimensional quasitoric manifold S with fiber $\mathbb{C}P^n$. Assume that $n = 1$ or $b_2(S) \leq 10$. Let M be a quasitoric manifold whose cohomology ring is isomorphic to $H^*(P(E))$. Then, M is homeomorphic to $P(E)$ unless $P(E)$ is a $\mathbb{C}P^1$ -bundle over $\mathbb{C}P^2 \# \mathbb{C}P^2$.*

The remainder of this paper is organized as follows. In Section 2, we define a projective bundle over a smooth compact toric surface. In Section 3, we introduce the definitions and properties of quasitoric manifolds, and we discuss a projective bundle as a quasitoric manifold. In Section 4, we show that the product of a simplex and a polygon is a combinatorially rigid polytope (the definition is given below). In Section 5, we show that if the cohomology ring of a quasitoric manifold is isomorphic to that of a projective bundle over a smooth compact toric surface S as a ring, then the quasitoric manifold also admits an equivariant bundle structure whose base space is diffeomorphic to S , except for the $\mathbb{C}P^1$ -bundle over $\mathbb{C}P^2$ case. Thus, Theorem 1.1 is proved. In Section 6, we classify a projective bundle over a smooth compact toric surface smoothly, and we also classify a quasitoric manifold that admits a projective bundle structure whose base space is a 4-dimensional quasitoric manifold. Further, the proofs of Theorem 1.2 and Theorem 1.3 are provided.

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2. PROJECTIVE BUNDLES OVER A SMOOTH COMPACT TORIC SURFACE

Let B be a smooth manifold, and let E be a complex vector bundle over B with the fiber space V . We can take the projectivization $P(E)$ by taking the projectivization of each fiber of E . Then, $P(E)$ is a fiber bundle over B with the fiber space $P(V)$.

Let x be the negative of the first Chern class of the tautological line bundle over $P(E)$. Then, the cohomology $H^*(P(E))$ of $P(E)$ can be regarded as an algebra over $H^*(B)$ via $\pi^*: H^*(B) \rightarrow H^*(P(E))$, where $\pi: P(E) \rightarrow B$ denotes the projection. When $H^*(B)$ is finitely generated and torsion free, π^* is injective, and $H^*(P(E))$ as an algebra over $H^*(B)$ is known to be described as

$$H^*(P(E)) = H^*(B)[x] \Big/ \left(\sum_{k=0}^n c_k(E)x^{n-k} \right),$$

where n is the complex dimension of V (see [1]).

Lemma 2.1. [8] *Let B and E be as above, and let L be a complex line bundle over B . Let E^* denote the complex vector bundle dual to E . Then, $P(E \otimes L)$, $P(E)$, and $P(E^*)$ are isomorphic as bundles over B ; in particular, they are diffeomorphic.*

Let S be a smooth compact toric surface. Then, S is obtained by blow-ups from either $\mathbb{C}P^2$ or one of the Hirzebruch surfaces (see [18]). In particular, S is diffeomorphic to $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, or the connected sum of $\mathbb{C}P^2$ with a finite number of copies of $\overline{\mathbb{C}P^2}$, where $\overline{\mathbb{C}P^2}$ denotes $\mathbb{C}P^2$ with reverse orientation (see [15]). Consequently, the cohomology $H^*(S)$ of S is finitely generated by the second cohomology classes, and it is torsion free. Hence, we may assume that $H^*(S)$ is generated by x_1, \dots, x_m of degree 2, where $m = \text{rank}_{\mathbb{Z}} H^2(S)$.

Now, let E be the Whitney sum of $n + 1$ complex line bundles over S . Then, $P(E)$ is a projective bundle over S with the fiber space $\mathbb{C}P^n$. By

Lemma 2.1, we may assume that

$$P(E) = P(\underline{\mathbb{C}} \oplus L_1 \oplus \cdots \oplus L_n),$$

where $\underline{\mathbb{C}}$ is the trivial complex line bundle and L_i 's are complex line bundles over S . Then, the cohomology $H^*(P(E))$ is a free module over $H^*(S)$ with basis $\{1, x, \dots, x^n\}$, and the ring structure is determined by the single relation

$$x^{n+1} + c_1(E)x^n + \cdots + c_n(E)x = 0,$$

where x is the negative of the first Chern class of the tautological line bundle over $P(E)$ and $c(E) = \prod_{i=1}^n c(L_i)$. Note that the first Chern class is a complete invariant for classifying complex line bundles smoothly. Hence, if the first Chern class of L_i is $c_1(L_i) := \sum_{j=1}^m a_{ij}x_j$ for each $i = 1, \dots, n$, then

$$c(E) = \prod_{i=1}^n \left(1 + \sum_{j=1}^m a_{ij}x_j \right).$$

Therefore, the cohomology ring of $P(E)$ is written as follows:

$$(2.1) \quad H^*(P(E)) = H^*(S)[x] \left/ x \prod_{i=1}^n \left(x + \sum_{j=1}^m a_{ij}x_j \right) \right.$$

Remark 2.2. Consider a vector bundle

$$U(3) \times_{S^1 \times U(2)} \mathbb{C}^2 \xrightarrow{\mathbb{C}^2} U(3)/(S^1 \times U(2)),$$

which cannot admit a splitting into the Whitney sum of complex line bundles. Since $U(3)/(S^1 \times U(2)) = \mathbb{C}P^2$, the projectivization $P(U(3) \times_{S^1 \times U(2)} \mathbb{C}^2)$ is a fiber bundle over $\mathbb{C}P^2$ with the fiber $\mathbb{C}P^1$. However, the projectivization of a complex vector bundle over a toric manifold is also a toric manifold if and only if the structural group of the complex vector bundle is diagonal. Hence, $P(U(3) \times_{S^1 \times U(2)} \mathbb{C}^2)$ is not a toric manifold.

We call $P(E)$ a *projective bundle over* a smooth manifold B only when E is the Whitney sum of complex line bundles over B . In particular, if $P(E)$ is a projective bundle over a smooth compact toric surface S , then it is also a projective toric manifold, as described in the introduction.

Example 2.3. Let $E_1 := \underline{\mathbb{C}} \oplus \bigoplus_{i=1}^n L_i$ and $E_2 := \underline{\mathbb{C}} \oplus \bigoplus_{i=1}^n L'_i$ be complex vector bundles over $S := \mathbb{C}P^2$, where L_i 's and L'_i 's are complex line bundles over S . Then, $P(E_1)$ and $P(E_2)$ are projective bundles over S with the fiber $\mathbb{C}P^n$. It is shown in [9] that

$$P(E_1) \approx P(E_2) \text{ if and only if } H^*(P(E_1)) \cong H^*(P(E_2)),$$

where \approx denotes “diffeomorphic” and \cong denotes “isomorphic as rings”.

This example shows that the cohomology ring determines the smooth type of a projective bundle $P(E)$ over $\mathbb{C}P^2$. Hence, we may ask the following question.

Problem 2.4. Are projective bundles $P(E_1)$ and $P(E_2)$ over a smooth compact toric surface S diffeomorphic or homeomorphic if $H^*(P(E_1)) \cong H^*(P(E_2))$ as rings?

We investigate this problem further in Section 6, where a partial affirmative answer is given.

3. QUASITORIC MANIFOLDS

Since a projective bundle $P(E)$ over a smooth compact toric surface is a projective toric manifold, it is also a quasitoric manifold. In this section, we recall general facts about quasitoric manifolds, and we consider a toric manifold $P(E)$ from a different point of view, i.e., as a quasitoric manifold.

Let M be a quasitoric manifold with an orbit map $\rho: M \rightarrow P$. Then, for a codimension- k face F of P , the preimage $\rho^{-1}(F)$ is a connected codimension- $2k$ submanifold of M , which is fixed pointwise by a k -dimensional subgroup of T^n . Let $\mathcal{F}(P) = \{F_1, \dots, F_d\}$ be the set of facets, codimension-one faces, of P . Then, there is a primitive vector λ_i in the integer lattice $\mathbb{Z}^n = \text{Hom}(S^1, T^n)$ of one-parameter circle subgroups in T^n such that λ_i spans the circle subgroup $T_{F_i} \subset T^n$ fixing the *characteristic submanifold* $\rho^{-1}(F_i)$. Hence, the vector λ_i is determined up to sign. Then, the map $\lambda: F_i \mapsto \lambda_i$ is called the *characteristic function* of M , and it satisfies the *non-singularity condition*:

$$(3.1) \quad \lambda(F_{i_1}), \dots, \lambda(F_{i_\alpha}) \text{ form a part of an integral basis of } \mathbb{Z}^n \\ \text{whenever the intersection } F_{i_1} \cap \dots \cap F_{i_\alpha} \text{ is non-empty.}$$

Let P be a simple polytope of dimension n and let $\mathcal{F}(P)$ be the set of facets of P . For a map $\lambda: \mathcal{F}(P) \rightarrow \mathbb{Z}^n$ satisfying the non-singularity condition (3.1), let T_F denote the subgroup of T^n represented by the unimodular subspace of \mathbb{Z}^n spanned by $\lambda(F_{i_1}), \dots, \lambda(F_{i_\alpha})$, where $F = F_{i_1} \cap \dots \cap F_{i_\alpha}$. Given a pair (P, λ) , we can construct a manifold

$$(3.2) \quad M(P, \lambda) := T^n \times P / \sim, \quad (t, p) \sim (s, q) \Leftrightarrow p = q \text{ and } t^{-1}s \in T_{F(p)},$$

where $F(p)$ is the face of P that contains a point $p \in P$ in its relative interior. Then, the standard T^n -action on T^n descends to a locally standard action of T^n on $M(P, \lambda)$ whose orbit space is combinatorially equivalent to P . Hence, $M(P, \lambda)$ is indeed a quasitoric manifold with the characteristic function λ .

It is shown in [13] that for a quasitoric manifold M over P with its characteristic function λ , there is an equivariant homeomorphism $M \rightarrow M(P, \lambda)$ covering the identity on P . Thus, any quasitoric manifold can be expressed by a pair of a simple polytope P and a map $\lambda: \mathcal{F}(P) \rightarrow \mathbb{Z}^n$ satisfying the non-singularity condition (3.1).

Note that one may assign an $n \times d$ matrix Λ , called a *characteristic matrix*, to a characteristic function λ by

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_d)) = (A|B),$$

where A is an $n \times n$ matrix and B is an $n \times (d - n)$ matrix. From the non-singularity condition (3.1), if we assume that $F_1 \cap \dots \cap F_n \neq \emptyset$, A is invertible. Hence, in this paper, for simplicity, we assume that the first n columns of Λ form an invertible matrix A , and we sometimes denote $M(P, \lambda)$ by $M(P, \Lambda)$ as long as there is no confusion.

We say that two quasitoric manifolds M_1 and M_2 over the same polytope P are *equivalent* if there is a θ -equivariant homeomorphism $f: M_1 \rightarrow M_2$

that covers the identity on P , where θ is an automorphism on T^n . Here, “ θ -equivariant homeomorphism f ” implies that the homeomorphism f satisfies $f(t \cdot x) = \theta(t) \cdot f(x)$ for all $t \in T^n$ and $x \in M$.

Assume that $M_1 = M(P, \lambda_1)$ and $M_2 = M(P, \lambda_2)$. If there is a general linear group $\sigma \in GL_n(\mathbb{Z})$ such that $\lambda_1 = \sigma \circ \lambda_2$, then M_1 and M_2 are equivalent. Hence, for each quasitoric manifold $M(P, \lambda)$, the corresponding matrix Λ can be represented by $(E_n | A^{-1}B)$, where E_n is the identity matrix of order n .

Let P_i be an n_i -dimensional simple polytope with $\mathcal{F}(P_i) = \{F_{i,1}, \dots, F_{i,d_i}\}$ for $i = 1, 2$. Then, the Cartesian product of two simple polytopes $P = P_1 \times P_2$ is an $(n_1 + n_2)$ -dimensional simple polytope with $(d_1 + d_2)$ facets. Note that each facet of P is the product of a facet of one and the other. For convenience, we shall give an order on $\mathcal{F}(P)$ by

$$\mathcal{F}(P) = \{F_1^1, \dots, F_{n_1}^1, F_1^2, \dots, F_{n_2}^2, F_{n_1+1}^1, \dots, F_{d_1}^1, F_{n_2+1}^2, \dots, F_{d_2}^2\},$$

where $F_j^1 = F_{1,j} \times P_2$ and $F_j^2 = P_1 \times F_{2,j}$. Now, let M be a quasitoric manifold over the polytope P , and set $n = n_1 + n_2$. Then, we obtain a characteristic function $\lambda : \mathcal{F}(P) \rightarrow \mathbb{Z}^n$. Up to equivalence, we may assume that the characteristic matrix Λ associated with λ is of the form

$$(3.3) \quad \Lambda = \begin{pmatrix} E_{n_1} & O & A_{11} & A_{12} \\ O & E_{n_2} & A_{21} & A_{22} \end{pmatrix},$$

where E_{n_i} is the identity matrix of order n_i and A_{ij} is an $n_i \times (d_j - n_j)$ matrix.

Lemma 3.1. $\Lambda_i := (E_{n_i}, A_{ii})$ is a characteristic matrix on P_i .

Proof. Note that for any vertex $v = F_{1,i_1} \cap \dots \cap F_{1,i_{n_1}}$ of P_1 ,

$$v \times (F_{2,1} \cap \dots \cap F_{2,n_2}) = F_{i_1}^1 \cap \dots \cap F_{i_{n_1}}^1 \cap F_1^2 \cap \dots \cap F_{n_2}^2$$

is also a vertex of P . Define $\lambda_1 : \mathcal{F}(P_1) \rightarrow \mathbb{Z}^{n_1}$ by mapping $\lambda_1(F_{1,j})$ to the j -th column vector of Λ_1 . Then,

$$\begin{aligned} & \det(\lambda_1(F_{1,i_1}), \dots, \lambda_1(F_{1,i_{n_1}})) \\ &= \det(\lambda(F_{i_1}^1), \dots, \lambda(F_{i_{n_1}}^1), \lambda(F_1^2), \dots, \lambda(F_{n_2}^2)) \\ &= \pm 1. \end{aligned}$$

Therefore, λ_1 satisfies the non-singularity condition on P_1 , and hence, Λ_1 is a characteristic matrix on P_1 . A similar argument shows that Λ_2 is also a characteristic matrix on P_2 . \square

Definition 3.2. Let M , B , and F be quasitoric manifolds of dimensions $2n$, $2m$, and $2n - 2m$, respectively. A bundle $\pi : M \rightarrow B$ with fiber F is said to be *equivariant* if there is a surjective homomorphism $\theta : T^n \rightarrow T^m$ such that $\pi(t \cdot x) = \theta(t) \cdot \pi(x)$ for all $t \in T^n$ and $x \in M$, the fiber $\pi^{-1}(b)$ has a locally standard action of $\ker \theta$ for each $b \in B$, and $\pi^{-1}(b)$ is equivalent to F .

If $M \rightarrow B$ is an equivariant bundle with fiber F , then the orbit space of M is the product of the orbit space of B and the orbit space of F by Proposition 5 in [14]. Furthermore, we have the following lemma, which is an immediate corollary of Theorem 6 in [14].

Lemma 3.3. *A quasitoric manifold M over $P_1 \times P_2$ is an equivariant bundle with fiber $M(P_1, \Lambda_1)$ and base $M(P_2, \Lambda_2)$ if and only if it is equivalent to $M(P_1 \times P_2, \Lambda)$, where Λ is the characteristic matrix of the form*

$$\begin{pmatrix} E_{n_1} & O & A_{11} & A_{12} \\ O & E_{n_2} & O & A_{22} \end{pmatrix}$$

and $\Lambda_i := (E_{n_i}, A_{ii})$ for $i = 1, 2$.

Let us consider a projective bundle $P(E)$ over a smooth compact toric surface S . We remark that S is a quasitoric manifold of (real) dimension 4 with a natural T^2 -action. Hence, the orbit space S/T^2 is a polygon. Assume that $P(E)$ is a $\mathbb{C}P^n$ -bundle over S . Then, $P(E)$ is also a quasitoric manifold, and its orbit space is $\Delta^n \times G(m+2)$, where Δ^n and $G(m+2)$ denote an n -simplex and an $(m+2)$ -gon, respectively.

Let us find the characteristic matrix of $P(E) = P(\mathbb{C} \oplus \bigoplus_{i=1}^n L_i)$ explicitly. Let $\rho: S \rightarrow G(m+2)$ be the orbit map, and let the characteristic matrix³ of S be given by

$$(3.4) \quad \Lambda_S = \begin{pmatrix} 1 & 0 & -\Lambda_1^1 & \cdots & -\Lambda_1^m \\ 0 & 1 & -\Lambda_2^1 & \cdots & -\Lambda_2^m \end{pmatrix},$$

where the order of facets of $G(m+2)$ is $F_{m+1}, F_{m+2}, F_1, \dots, F_m$. Since S is a projective toric manifold, it has a T^2 -invariant complex structure. Thus, every normal bundle $\nu_j := \nu(S_j \subset S)$ over a characteristic submanifold $S_j := \rho^{-1}(F_j)$, $j = 1, \dots, m+2$, is a T^2 -invariant complex line bundle. Let γ_j be the complex T^2 -bundle over S extending the normal bundle ν_j of the characteristic submanifold S_j trivially outside the tubular neighborhood of S_j . Then, the first Chern class $c_1(\gamma_j) \in H^2(S)$ is dual to the characteristic submanifold S_j for $j = 1, \dots, m+2$. Since $c_1(\gamma_1), \dots, c_1(\gamma_m)$ generate $H^2(S)$ and every complex line bundle is classified by its first Chern class, each complex line bundle L_i over S is isomorphic to $\gamma_1^{a_{i1}} \otimes \cdots \otimes \gamma_m^{a_{im}}$ for some integers a_{i1}, \dots, a_{im} . Hence, $P(E)$ is isomorphic to $P(\mathbb{C} \oplus \bigoplus_{i=1}^n (\bigotimes_{j=1}^m \gamma_j^{a_{ij}}))$ for some integers $a_{ij} \in \mathbb{Z}$.

Proposition 3.4. *Let S and γ_j be given as above. Let $L_i = \bigotimes_{j=1}^m \gamma_j^{a_{ij}}$ for $i = 1, \dots, n$. Then, $M = P(\mathbb{C} \oplus \bigoplus_{i=1}^n L_i)$ is a quasitoric manifold whose characteristic matrix is of the form*

$$(3.5) \quad \Lambda = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & 0 & 0 & 0 & -1 & -a_{11} & \cdots & -a_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & -1 & -a_{n,1} & \cdots & -a_{n,m} \\ \hline 0 & \cdots & 0 & 1 & 0 & 0 & -\Lambda_1^1 & \cdots & -\Lambda_1^m \\ 0 & \cdots & 0 & 0 & 1 & 0 & -\Lambda_2^1 & \cdots & -\Lambda_2^m \end{array} \right),$$

where the order of facets of $\Delta^n \times G(m+2)$ is

$$F_1^1, \dots, F_n^1, F_{m+1}^2, F_{m+2}^2, F_{n+1}^1, F_1^2, \dots, F_m^2.$$

³Since S is a projective toric manifold, the columns of Λ_S generate the normal fan of a polygon in \mathbb{R}^2 . That is to say that the polygon $\{\mathbf{x} \in \mathbb{R}^2 \mid \langle \mathbf{x}, \boldsymbol{\lambda}_S(F_i) \rangle \geq 0 \text{ for all } i = 1, \dots, m+2\}$ is identified with the orbit space of S .

Proof. Let $X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$ for a topological space X with a group action of G . Note that for a fixed point $p \in S^{T^2}$, we know that

- $\gamma_i|_p = \nu_i|_p$ as T^2 -modules if $p \in S_i^{T^2}$, and
- $\gamma_i|_p$ is the trivial T^2 -module if $p \notin S_i^{T^2}$.

Remark that $\lambda_S(F_j)(S^1) \subset T^2$ is the circle subgroup that fixes $S_j \subset S$. Since the T^2 -action on S is effective, the action of $\lambda_S(F_j)(s)$ ($s \in S^1$) on $\gamma_i|_{S_j}$ is the complex multiplication on fibers by $s \in S^1 \subset \mathbb{C}$ when $i = j$, and trivial if $i \neq j$.

On the other hand, we can give an additional T^n -action on a complex T^2 -bundle $\underline{\mathbb{C}} \oplus \bigoplus_{i=1}^n L_i$ by

$$(3.6) \quad (t_1, \dots, t_n) \cdot (u_0, u_1, \dots, u_n) = (u_0, t_1 u_1, \dots, t_n u_n).$$

Then, the total space of the projective bundle $\pi: M = P(\underline{\mathbb{C}} \oplus \bigoplus_{i=1}^n L_i) \rightarrow S$ has a T^{n+2} -action and is a quasitoric manifold with characteristic submanifolds $M_j := \pi^{-1}(S_j)$ for $j = 1, \dots, m+2$ and $N_r := \{u_r = 0\}$ for $r = 0, 1, \dots, n$.

To find the characteristic matrix of M , we need to know which circle subgroup of T^{n+2} fixes each characteristic submanifold. The r -th component of T^{n+2} fixes N_r when $r = 1, \dots, n$, and the circle subgroup

$$\{(t^{-1}, \dots, t^{-1}, 1, 1) \in T^{n+2} \mid t \in S^1\} \subset T^{n+2}$$

fixes N_0 . The action of $\lambda_S(F_j)(s)$ ($s \in S^1$) on fibers of $L_i|_{S_j}$ is the complex multiplication by $s^{a_{ij}}$ when $1 \leq j \leq m$, and is trivial when $j = m+1, m+2$, while the action of (t_1, \dots, t_n) on L_i is the complex multiplication by t_i by (3.6). Therefore, in order to make the fiberwise action trivial, we have

$$(t_1, \dots, t_n) = (s^{-a_{1j}}, \dots, s^{-a_{nj}})$$

when $1 \leq j \leq m$ and

$$(t_1, \dots, t_n) = (1, \dots, 1)$$

when j is $m+1$ or $m+2$. Thus, the proposition is proved. \square

If S is a 4-dimensional quasitoric manifold that is not a smooth compact toric surface, then it does not necessarily admit a T^2 -invariant almost complex structure. However, in this case, we can give an additional structure on S , the *omniorientation*, which is defined to be a choice of an orientation for S and of orientations for each of characteristic submanifolds S_i , $i = 1, \dots, m+2$. Then, the omniorientation determines an orientation for every normal bundle ν_i . Since every ν_i is a two-plane bundle, it follows that an orientation of ν_i enables us to interpret ν_i as a complex line bundle. Since the torus T^2 is oriented, choosing an orientation for $G(m+2)$ is equivalent to choosing an orientation for S . Since each circle subgroup T_{F_i} fixing S_i acts on the normal bundle ν_i , a choice of an omniorientation for S is equivalent to a choice of an orientation for $G(m+2)$ together with an unambiguous choice of column vectors Λ_S^i .

From a similar argument to the proof of Proposition 3.4, we obtain the following corollary.

Corollary 3.5. *If S is a 4-dimensional quasitoric manifold with an omniorientation, then there is a projective bundle over S whose characteristic matrix is of the form (3.5).*

We close this section with a brief review of the cohomology ring of a quasitoric manifold. Let P be an n -dimensional simple polytope with d facets and $\mathcal{F}(P) = \{F_1, \dots, F_d\}$. Let $\mathbf{k}[v_1, \dots, v_d]$ denote the polynomial ring in d variables over a commutative ring \mathbf{k} with unit, $\deg v_i = 2$. We primarily assume that \mathbf{k} is a ring of integers \mathbb{Z} or a ring of rational numbers \mathbb{Q} . We identify each $F_i \in \mathcal{F}(P)$ with the indeterminate v_i . The *face ring* (or *Stanley-Reisner ring*) $\mathbf{k}(P)$ of P is the quotient ring

$$\mathbf{k}(P) = \mathbf{k}[v_1, \dots, v_d]/I_P,$$

where I_P is the ideal generated by the monomials $v_{i_1} \cdots v_{i_\ell}$ whenever $F_{i_1} \cap \cdots \cap F_{i_\ell} = \emptyset$. The ideal I_P is called the *Stanley-Reisner ideal* of P .

Let M be a quasitoric manifold over P with the projection $\rho: M \rightarrow P$ and the characteristic function λ . Then, one can find an isomorphism between the face ring $\mathbb{Z}(P)$ and the equivariant cohomology ring of M with \mathbb{Z} coefficients:

$$H_T^*(M) \cong \mathbb{Z}[v_1, \dots, v_d]/I_P = \mathbb{Z}(P),$$

where v_j is the equivariant Poincaré dual of the codimension-two invariant submanifold $M_j = \rho^{-1}(F_j)$ in M . Note that $H_T^*(M)$ is not only a ring but also an $H^*(BT) = \mathbb{Z}[t_1, \dots, t_n]$ -module via the map p^* , where $p: ET \times_T M \rightarrow BT$ is the natural projection, and p^* takes t_i to $\theta_i := \lambda_{i1}v_1 + \cdots + \lambda_{id}v_d \in \mathbb{Z}(P)$, where $\lambda(F_i) = (\lambda_{1i}, \dots, \lambda_{ni})^T \in \mathbb{Z}^n$ for $i = 1, \dots, n$. Since everything has vanishing odd degrees, $H_T^*(M)$ is a free $H^*(BT)$ -module. Hence, the kernel of $\mathbb{Z}(P) = H_T^*(M) \rightarrow H^*(M)$ is the ideal J_λ of $\mathbb{Z}(P)$ generated by $\theta_1, \dots, \theta_n$. Therefore, we have

$$(3.7) \quad H^*(M) = \mathbb{Z}[v_1, \dots, v_d]/(I_P + J_\lambda).$$

See [13] for more details of the previous argument.

4. COMBINATORIAL RIGIDITY FOR THE PRODUCT OF A SIMPLEX AND A POLYGON

In this section, we claim that a quasitoric manifold whose cohomology ring is isomorphic to that of a projective bundle $P(E)$ has the orbit space combinatorially equivalent to $\Delta^n \times G(m+2)$. In order to do this, we introduce one of important invariants of a simple polytope coming from its face ring.

Let P be a simple polytope with d facets. A *finite free resolution* $[R : \phi]$ of a face ring $\mathbb{Q}(P)$ is an exact sequence

$$(4.1) \quad 0 \longrightarrow R^{-r} \xrightarrow{\phi} R^{-r+1} \xrightarrow{\phi} \cdots \xrightarrow{\phi} R^{-1} \xrightarrow{\phi} R^0 \xrightarrow{\phi} \mathbb{Q}(P) \longrightarrow 0,$$

where R^{-i} is a finite free $\mathbb{Q}[x_1, \dots, x_d]$ -module and each differential map ϕ is degree-preserving. If we take R^{-i} to be the module generated by the minimal basis of $\text{Ker}(\phi)$, we get a *minimal resolution* of $\mathbb{Q}(P)$. Since $\mathbb{Q}(P)$ is graded, so are all R^{-i} 's, that is, $R^{-i} = \bigoplus_j R^{-i, 2j}$. Let

$$\beta^{-i, 2j}(P) = \dim_{\mathbb{Q}} R^{-i, 2j},$$

and we call it the $(-i, 2j)$ -th *bigraded Betti number* of P .

Theorem 4.1. [19] *Let P be a simple polytope with facets F_1, \dots, F_d . Then, we have*

$$\beta^{-i, 2j}(P) = \sum_{\substack{|\sigma|=j, \\ \sigma \subset \{1, \dots, d\}}} \dim \tilde{H}_{j-i-1}(\bigcup_{i \in \sigma} F_i).$$

Here, $\dim \tilde{H}_{-1}(\emptyset) = 1$ by convention.

Bigraded Betti numbers also satisfy the following relations (see [3] for details).

Proposition 4.2. *Let P be an n -dimensional simple polytope with d facets. Then, we have the following:*

- (1) $\beta^{0,0}(P) = \beta^{-(d-n), 2d}(P) = 1$;
- (2) (Poincaré duality) $\beta^{-i, 2j}(P) = \beta^{-(d-n)+i, 2(d-j)}(P)$;
- (3) $\beta^{-i, 2j}(P_1 \times P_2) = \sum_{\substack{i'+i''=i \\ j'+j''=j}} \beta^{-i', 2j'}(P_1) \beta^{-i'', 2j''}(P_2)$.

Example 4.3 (Bigraded Betti numbers of the product of a simplex and a polygon). Since every proper subset of facets of the n -simplex Δ^n is contractible, by Theorem 4.1, we have that for all i, j ,

$$\beta^{-i, 2j}(\Delta^n) = 0, \quad \text{except for } \beta^{0,0}(\Delta^n) = \beta^{-1, 2(n+1)}(\Delta^n) = 1.$$

Using Theorem 4.1, it is a good exercise to prove that

$$\beta^{-i, 2j}(G(m+2)) = \begin{cases} 1, & \text{if } (i, j) = (0, 0), (m, m+2); \\ \frac{(m+2)(k-1)}{m+2-k} \binom{m}{k}, & \text{if } (i, j) = (k-1, k); \\ 0, & \text{otherwise} \end{cases}$$

(see [4, Corollary 3.7]). Assume that $n > 1$. Then, by Proposition 4.2 (3), we can compute the bigraded Betti numbers of $\Delta^n \times G(m+2)$, which is an $n+2$ dimensional simple polytope having $m+n+3$ facets:

$$\beta^{-i, 2j}(\Delta^n \times G(m+2)) = \begin{cases} 1, & \text{if } (i, j) = (0, 0), (1, n+1), \\ & (m, m+2), (m+1, m+n+3); \\ \frac{(m+2)(k-1)}{m+2-k} \binom{m}{k}, & \text{if } (i, j) = (k-1, k), (k, k+n+1); \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.4. A simple polytope P is (toric) *cohomologically rigid* if there exists a quasitoric manifold M over P , and whenever there exists a quasitoric manifold N over a simple polytope Q with a graded ring isomorphism $H^*(M) \cong H^*(N)$, Q is combinatorially equivalent to P .

Since Choi-Panov-Suh [10] showed that for two quasitoric manifolds M and N over P and Q , respectively, $H^*(M) \cong H^*(N)$ implies that $\beta^{-i, 2j}(P) = \beta^{-i, 2j}(Q)$ for all i and j , one efficient way to decide the cohomological rigidity of a simple polytope P is to check the uniqueness of its bigraded Betti numbers among all simple polytopes.

Definition 4.5. A simple polytope P is (toric) *combinatorially rigid* if Q is combinatorially equivalent to P whenever $\beta^{-i, 2j}(Q) = \beta^{-i, 2j}(P)$ for all i, j .

We note that if P supports a quasitoric manifold and it is combinatorially rigid, then P is cohomologically rigid.

Theorem 4.6. *A product of a simplex and a polygon is combinatorially rigid, that is, if a simple polytope P satisfies*

$$\beta^{-i,2j}(P) = \beta^{-i,2j}(\Delta^n \times G(m+2))$$

for all i and j , then P is combinatorially equivalent to $\Delta^n \times G(m+2)$.

Proof. When $m \leq 2$, $G(m+2)$ is either Δ^2 or $(\Delta^1)^2$. Since the product of simplices is combinatorially rigid by [10], the assertion is true.

When $n = 1$, $\Delta^1 \times G(m+2)$ is an $(m+2)$ -gonal prism. From [5], it is also known that any prism is combinatorially rigid. Therefore, the assertion is also true for this case.

Now, assume that $m > 2$ and $n > 1$. Since the bigraded Betti numbers determine the dimension and the number of facets of the polytope, from the assumption, P is a simple polytope of dimension $n+2$ with $m+n+3$ facets and, by Example 4.3,

$$\beta^{-1,2j}(P) = \begin{cases} 1, & \text{if } j = n+1; \\ \frac{(m+2)(m-1)}{2}, & \text{if } j = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\beta^{-1,2(n+1)}(P) = 1$, there is a set W of $n+1$ facets that satisfies

$$\dim_{\mathbb{Q}} \tilde{H}_{n-1}(\bigcup_{F \in W} F) = 1.$$

Such a set is unique, and we can put $W = \{F_{m+3}, \dots, F_{m+n+3}\} \subset \mathcal{F}(P)$. It follows from Alexander duality that the complement $W^c \subset \mathcal{F}(P)$, say $W^c = \{F_1, \dots, F_{m+2}\}$, has the same homology groups as the circle S^1 . Hence, we have

$$\dim_{\mathbb{Q}} \tilde{H}_1(\bigcup_{F \in W^c} F) = 1.$$

Moreover, since $\beta^{-j,2(n+j)}(P) = 0$ for all $j > 1$, by Proposition 4.2 (2), we have $\beta^{-(m-j+1),2(m-j+3)}(P) = 0$, that is, any union of $m-j+3$ facets cannot be homotopy equivalent to S^1 . In other words, by the appropriate re-indexing of facets of W^c , F_i intersects with exactly two facets F_{i-1} and F_{i+1} for all $i = 1, \dots, m+2$, where the indices are up to modulo $m+2$.

We note that $\beta^{-1,2j}(P)$ is the number of monomial generators of degree j for I_P , the Stanley-Reisner ideal of P . Since

$$\begin{aligned} \beta^{-1,4}(P) &= \sum_{|V|=2, V \subset \mathcal{F}(P)} \tilde{H}_0(\bigcup_{F \in V} F) \\ &\geq \sum_{|V|=2, V \subset W^c} \tilde{H}_0(\bigcup_{F \in V} F) \\ &= \frac{(m+2)(m-1)}{2} = \beta^{-1,4}(P), \end{aligned}$$

we can deduce that there are no generators for I_P that are divisible by $x_k x_{m+2+j}$ for some $1 \leq k \leq m+2$ and $1 \leq j \leq n+1$, where x_i is the indeterminate corresponding to F_i . Hence, the face ring $\mathbb{Q}(P)$ of P can be decomposed as

$$\mathbb{Q}(P) = \mathbb{Q}[x_1, \dots, x_{m+2}]/I \otimes \mathbb{Q}[x_{m+3}, \dots, x_{m+n+3}]/\langle x_{m+3} \cdots x_{m+n+3} \rangle,$$

where I is the ideal generated by $x_i x_j$'s with $j \not\equiv i \pm 1 \pmod{m+2}$. Hence,

$$\mathbb{Q}(P) = \mathbb{Q}(G(m+2)) \otimes \mathbb{Q}(\Delta^n) = \mathbb{Q}(\Delta^n \times G(m+2)).$$

Since the face ring completely determines the combinatorial type of the polytope by [2], P is combinatorially equivalent to $\Delta^n \times G(m+2)$. \square

Corollary 4.7. *If the cohomology ring of a quasitoric manifold M is isomorphic to that of a projective bundle $P(E)$, then the orbit space of M is combinatorially equivalent to that of $P(E)$.*

5. COHOMOLOGY DETERMINES AN EQUIVARIANT BUNDLE STRUCTURE

We remark that a (generalized) Bott manifold admits an iterated equivariant bundle structure. It was shown in [11] that when the cohomology ring of a quasitoric manifold M is isomorphic to that of a two-stage generalized Bott manifold B , the quasitoric manifold is homeomorphic to B . Furthermore, if the dimension of fiber is not equal to 1, then M also admits an equivariant bundle structure. On the other hand, remarkably, it was shown in [12] that if $H^*(M)$ is isomorphic to the cohomology ring of some Bott manifold, then M should admit an iterated equivariant bundle structure. Hence, we can conclude that the cohomology ring of a quasitoric manifold should have the information of an equivariant bundle structure.

In this section, we claim that a quasitoric manifold whose cohomology ring is isomorphic to that of a projective bundle over a smooth compact toric surface must admit an equivariant bundle structure, excluding the case of a $\mathbb{C}P^1$ -bundle over $\mathbb{C}P^2$.

Theorem 5.1. *Let $P(E)$ be a projective bundle over a smooth compact toric surface S with fiber $\mathbb{C}P^n$. Let M be a quasitoric manifold whose cohomology ring is isomorphic to $H^*(P(E))$. Then, M is*

- (1) *homeomorphic to a two-stage generalized Bott manifold when $n = b_2(S) = 1$, or*
- (2) *equivalent to a fiber bundle over a 4-dimensional quasitoric manifold with the fiber space $\mathbb{C}P^n$ when $n \geq 2$ or $b_2(S) \geq 2$.*

In particular, if M is a toric manifold, then M is equivalent to a projective bundle over a smooth compact toric surface.

Proof. Let $b_2(S) = m$ and $H^*(S) = \mathbb{Z}[y_1, \dots, y_m]/\mathcal{I}_S$, where \mathcal{I}_S is the ideal generated by $y_k y_\ell$ for $k \neq \ell \pm 1$, $y_i(\Lambda_1'^1 y_1 + \Lambda_1'^2 y_2 + \dots + \Lambda_1'^m y_m)$ for $i = 1, \dots, m-1$, and $y_j(\Lambda_2'^1 y_1 + \Lambda_2'^2 y_2 + \dots + \Lambda_2'^m y_m)$ for $j = 2, \dots, m$. By (2.1), we may assume that

$$\begin{aligned} H^*(P(E)) &= H^*(S)[y_0] \Big/ y_0 \prod_{i=1}^n \left(y_0 + \sum_{j=1}^m a'_{ij} y_j \right) \\ &= \mathbb{Z}[y_0, y_1, \dots, y_m]/\mathcal{I}', \end{aligned}$$

where \mathcal{I}' is the ideal generated by a monomial

$$(5.1) \quad y_0 \prod_{i=1}^n \left(y_0 + \sum_{j=1}^m a'_{ij} y_j \right)$$

and the ideal \mathcal{I}_S .

By Corollary 4.7, the orbit space of M is $\Delta^n \times G(m+2)$. Thus, we may assume that the characteristic matrix of M is of the form

$$(5.2) \quad \Lambda = \left(\begin{array}{ccc|cc|c|ccc} 1 & \cdots & 0 & 0 & 0 & -1 & -a_{1,1} & \cdots & -a_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & -1 & -a_{n,1} & \cdots & -a_{n,m} \\ \hline 0 & \cdots & 0 & 1 & 0 & -b & -\Lambda_1^1 & \cdots & -\Lambda_1^m \\ 0 & \cdots & 0 & 0 & 1 & -c & -\Lambda_2^1 & \cdots & -\Lambda_2^m \end{array} \right),$$

up to equivalence⁴. Then, the cohomology ring of M is

$$H^*(M) = \mathbb{Z}[x_0, x_1, \dots, x_m]/\mathcal{I},$$

where \mathcal{I} is the ideal generated by polynomials

$$\begin{aligned} & x_0 \prod_{i=1}^n (x_0 + \sum_{j=1}^m a_{i,j} x_j), \\ & x_i (bx_0 + \Lambda_1^1 x_1 + \Lambda_1^2 x_2 + \cdots + \Lambda_1^m x_m) \quad \text{for } i = 1, \dots, m-1, \\ & x_j (cx_0 + \Lambda_2^1 x_1 + \Lambda_2^2 x_2 + \cdots + \Lambda_2^m x_m) \quad \text{for } j = 2, \dots, m, \text{ and} \\ & x_k x_\ell \quad \text{for } k \neq \ell \pm 1. \end{aligned}$$

Furthermore, the matrix

$$\Lambda_N = \begin{pmatrix} 1 & 0 & -\Lambda_1^1 & \cdots & -\Lambda_1^m \\ 0 & 1 & -\Lambda_2^1 & \cdots & -\Lambda_2^m \end{pmatrix}$$

is a characteristic matrix on $G(m+2)$ by Lemma 3.1.

Note that if $b = c = 0$, then the characteristic matrix of M is of the form (3.5), but $M(G(m+2), \Lambda_N)$ is just a 4-dimensional quasitoric manifold (not necessarily a smooth compact toric surface in general), and then, M is equivalent to a quasitoric manifold that is a fiber bundle over a 4-dimensional quasitoric manifold with the fiber space $\mathbb{C}P^n$ by Lemma 3.1.

From the hypothesis, there is a graded ring isomorphism $\psi: H^*(M) \rightarrow H^*(P(E))$. Then, ψ lifts to a grading preserving isomorphism

$$\bar{\psi}: \mathbb{Z}[x_0, x_1, \dots, x_m] \rightarrow \mathbb{Z}[y_0, y_1, \dots, y_m]$$

with $\bar{\psi}(\mathcal{I}) = \mathcal{I}'$. Then, we can obtain a matrix $\mathbf{P} = [P_i^j]$, $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, m$, such that

$$\begin{pmatrix} \bar{\psi}(x_0) \\ \bar{\psi}(x_1) \\ \vdots \\ \bar{\psi}(x_m) \end{pmatrix} = \begin{pmatrix} P_0^0 & P_0^1 & \cdots & P_0^m \\ P_1^0 & P_1^1 & \cdots & P_1^m \\ \vdots & \vdots & \ddots & \vdots \\ P_m^0 & P_m^1 & \cdots & P_m^m \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}$$

and $\det(\mathbf{P}) = \pm 1$. We consider two cases (1) $n \geq 2$ and (2) $n = 1$, separately.

Case 1: $n \geq 2$. For each $k = 1, \dots, m-1$, since $x_k(bx_0 + \Lambda_1^1 x_1 + \Lambda_1^2 x_2 + \cdots + \Lambda_1^m x_m)$ is quadratic and $y_0 \prod_{i=1}^n (y_0 + \sum_{j=1}^m a'_{i,j} y_j)$ is not quadratic, $y_0 \prod_{i=1}^n (y_0 + \sum_{j=1}^m a'_{i,j} y_j)$ does not appear in $\bar{\psi}(x_k(bx_0 + \Lambda_1^1 x_1 + \Lambda_1^2 x_2 + \cdots + \Lambda_1^m x_m)) \in \mathcal{I}'$ as a component of its linear combination. Hence, $P_k^0 = 0$ for $k = 1, \dots, m-1$. Similarly, since $x_m(cx_0 + \Lambda_2^1 x_1 + \Lambda_2^2 x_2 + \cdots + \Lambda_2^m x_m)$ is also

⁴From the non-singularity condition, we obtain a column vector $(\pm 1, \dots, \pm 1, -b, -c)^t$ for the $(n+3)$ -th column of (5.2). Since the matrix that changes the row of (5.2) is an element of $GL_{n+2}(\mathbb{Z})$ and a characteristic function is determined up to sign, we can change all $+1$'s in the $(n+3)$ -th column to -1 up to equivalence.

quadratic, we obtain $P_m^0 = 0$. Therefore, $P_k^0 = 0$ for all $k = 1, \dots, m$. Note that $y_0 \prod_{i=1}^n (y_0 + \sum_{j=1}^m a'_{ij} y_j)$ is only one generator containing y_0 among the generators of \mathcal{I}' . This implies that $b = c = 0$.

Case 2: $n = 1$. If $m = 1$, $P(E)$ is a two-stage generalized Bott manifold; hence, M is homeomorphic to a two-stage generalized Bott manifold by Corollary 5.6 in [11].

If $m = 2$, $P(E)$ is a $\mathbb{C}P^1$ -bundle over a Hirzebruch surface. Since a Hirzebruch surface is a two-stage Bott manifold, $P(E)$ is a three-stage Bott manifold. Hence, M is equivalent to a three-stage Bott manifold by [12].

Now, we consider the case $m \geq 3$. If $F_i^2 \cap F_j^2 = \emptyset$, then $x_i x_j \in \mathcal{I}$ and we have

$$\bar{\psi}(x_i x_j) = (P_i^0 y_0 + P_i^1 y_1 + \dots + P_i^m y_m)(P_j^0 y_0 + P_j^1 y_1 + \dots + P_j^m y_m).$$

Note that if a quadratic element in \mathcal{I}' contains a term divisible by y_0 , the exponent of y_0 of the term must be equal to 2. Hence, we can see that $\bar{\psi}(x_i x_j) \in \mathcal{I}'$ only if either $P_i^0 = P_j^0 = 0$ or $P_i^0 P_j^0 \neq 0$. Hence, by considering the elements $x_1 x_3, x_1 x_4, \dots, x_1 x_m$ in \mathcal{I} , the entries $P_1^0, P_3^0, \dots, P_m^0$ are either all zero or all nonzero. Since $m \geq 3$, the four polynomials $x_1(bx_0 + \Lambda_1^1 x_1 + \dots + \Lambda_1^m x_m)$, $x_2(bx_0 + \Lambda_2^1 x_1 + \dots + \Lambda_2^m x_m)$, $x_2(cx_0 + \Lambda_2^1 x_1 + \dots + \Lambda_2^m x_m)$, and $x_m(cx_0 + \Lambda_m^1 x_1 + \dots + \Lambda_m^m x_m)$ belong to \mathcal{I} . Therefore, similarly, their images via $\bar{\psi}$ belong to \mathcal{I}' only if $P_1^0, P_2^0, P_m^0, bP_0^0 + \Lambda_1^1 P_1^0 + \dots + \Lambda_1^m P_m^0$, and $cP_0^0 + \Lambda_2^1 P_1^0 + \dots + \Lambda_2^m P_m^0$ are either all zero or all nonzero. One can observe that P_1^0, \dots, P_m^0 are either all zero or all nonzero.

Suppose that P_1^0, \dots, P_m^0 are all nonzero. Then, both $bP_0^0 + \Lambda_1^1 P_1^0 + \dots + \Lambda_1^m P_m^0$ and $cP_0^0 + \Lambda_2^1 P_1^0 + \dots + \Lambda_2^m P_m^0$ are also nonzero. For the pair (i, j) with $j \neq i \pm 1$, we have

$$(5.3) \quad \begin{aligned} \bar{\psi}(x_i x_j) &= P_i^0 P_j^0 y_0^2 + \sum_{\ell=1}^m (P_i^\ell P_j^0 + P_i^0 P_j^\ell) y_0 y_\ell \\ &\quad + (P_i^1 y_1 + \dots + P_i^m y_m)(P_j^1 y_1 + \dots + P_j^m y_m). \end{aligned}$$

Since $\bar{\psi}(x_1 x_j) \in \mathcal{I}'$ for $j = 3, \dots, m$, from (5.1) and (5.3), we obtain

$$P_1^0 P_j^0 a'_{1,\ell} = P_1^\ell P_j^0 + P_1^0 P_j^\ell$$

for $\ell = 1, \dots, m$. Since $P_1^0 \neq 0$, we can see that

$$(5.4) \quad P_j^\ell = \left(a'_{1,\ell} - \frac{P_1^\ell}{P_1^0} \right) P_j^0$$

for $\ell = 1, \dots, m$. Since $\bar{\psi}(x_1(bx_0 + \Lambda_1^1 x_1 + \dots + \Lambda_1^m x_m)) \in \mathcal{I}'$, we also obtain

$$\begin{aligned} &P_1^0 (bP_0^0 + \Lambda_1^1 P_1^0 + \dots + \Lambda_1^m P_m^0) a'_{1,\ell} \\ &= P_1^\ell (bP_0^0 + \Lambda_1^1 P_1^0 + \dots + \Lambda_1^m P_m^0) + P_1^0 (bP_0^\ell + \Lambda_1^1 P_1^\ell + \dots + \Lambda_1^m P_m^\ell), \end{aligned}$$

and hence, we can see that

$$(5.5) \quad bP_0^\ell + \Lambda_1^1 P_1^\ell + \dots + \Lambda_1^m P_m^\ell = \left(a'_{1,\ell} - \frac{P_1^\ell}{P_1^0} \right) (bP_0^0 + \Lambda_1^1 P_1^0 + \dots + \Lambda_1^m P_m^0).$$

Substituting $P_j^\ell = \left(a'_{1,\ell} - \frac{P_1^\ell}{P_1^0}\right) P_j^0$ into (5.5) for $j = 3, \dots, m$, we have

$$(5.6) \quad bP_0^\ell + \Lambda_1^1 P_1^\ell + \Lambda_1^2 P_2^\ell = \left(a'_{1,\ell} - \frac{P_1^\ell}{P_1^0}\right) (bP_0^0 + \Lambda_1^1 P_1^0 + \Lambda_1^2 P_2^0)$$

for $\ell = 1, \dots, m$.

From (5.4) and (5.6), we have

$$P_3^0(b\mathbf{P}_0 + \Lambda_1^1 \mathbf{P}_1 + \Lambda_1^2 \mathbf{P}_2) = (bP_0^0 + \Lambda_1^1 P_1^0 + \Lambda_1^2 P_2^0) \mathbf{P}_3,$$

where \mathbf{P}_i is the i -th row vector of $\mathbf{P} = [P_i^j]$. This implies that the first four row vectors of the matrix \mathbf{P} are linearly dependent, which contradicts the claim that ψ is an isomorphism. Therefore, $P_i^0 = 0$ for $i = 1, \dots, m$ and $P_0^0 \neq 0$. At the same time, both $bP_0^0 + \Lambda_1^1 P_1^0 + \dots + \Lambda_1^m P_m^0$ and $cP_0^0 + \Lambda_2^1 P_1^0 + \dots + \Lambda_2^m P_m^0$ are zero. Therefore, $b = c = 0$, which proves the former part of the theorem by Lemma 3.3.

Now, suppose that M is a toric manifold. If $n = m = 1$, M is a two-stage generalized Bott manifold by [9]. Otherwise, by the above argument, $b = c = 0$. Hence, the 4-dimensional quasitoric manifold $M(G(m+2), \Lambda_N)$ is indeed a smooth compact toric surface since M has a T^{n+2} -invariant almost complex structure. Therefore, M is equivalent to a projective bundle over a smooth compact toric surface, which proves the latter part of the theorem. \square

The following example shows that when $n = b_2(S) = 1$, there is a quasitoric manifold whose cohomology ring is isomorphic to $H^*(P(E))$, but which cannot admit an equivariant bundle structure.

Example 5.2. A quasitoric manifold over $\Delta^n \times \Delta^1$ is determined by a pair $(a, \mathbf{b}) \in \mathbb{Z} \times \mathbb{Z}^n$, where $\mathbf{b} = (b_1, \dots, b_n)$ and $1 - ab_i = \pm 1$ for all $i = 1, \dots, n$. Denote such a quasitoric manifold by $M_{a,\mathbf{b}}$. If a projective bundle $P(E)$ is a \mathbb{CP}^1 -bundle over \mathbb{CP}^n , then $P(E) = M_{a,\mathbf{0}}$ for some $a \in \mathbb{Z}$. If the number of nonzero ab_i 's is even, then $H^*(M_{a,\mathbf{b}})$ is isomorphic to $H^*(M_{a,\mathbf{0}})$. Moreover, $M_{a,\mathbf{b}}$ is homeomorphic to $M_{a,\mathbf{0}} = P(E)$. For example, $M_{2,(1,1)}$ is homeomorphic to $M_{2,(0,0)}$ (see [11] for further details).

Lemma 5.3. *Let $M = P(E)$ and $M' = P(E')$ be toric manifolds that are projective bundles over smooth compact toric surfaces S and S' , respectively. Then, a graded ring isomorphism from $H^*(M)$ to $H^*(M')$ induces a graded ring isomorphism from $H^*(S)$ to $H^*(S')$ provided that (i) $n \geq 2$ & $b_2(S) \geq 2$ or (ii) $n = 1$ & $b_2(S) \geq 3$.*

Proof. By Theorem 4.6, the orbit spaces of M and M' are combinatorially equivalent to the same polytope, a product of a simplex and a polygon, and the number of edges of the polygon is determined by $b_2(S)$ (or $b_2(S')$). Hence, $b_2(S) = b_2(S')$, say m . By (2.1), we may assume that

$$H^*(P(E)) = H^*(S)[x_0] \left/ y_0 \prod_{i=1}^n \left(x_0 + \sum_{j=1}^m a_{ij} x_j \right) \right.,$$

and

$$H^*(P(E')) = H^*(S')[y_0] \left/ y_0 \prod_{i=1}^n \left(y_0 + \sum_{j=1}^m a'_{ij} y_j \right) \right.$$

Let ψ be a graded ring isomorphism from $H^*(M)$ to $H^*(M')$, and let $\bar{\psi}$ be the induced grading preserving isomorphism from $\mathbb{Z}[x_0, x_1, \dots, x_m]$ to $\mathbb{Z}[y_0, y_1, \dots, y_m]$. Then, $\bar{\psi}$ induces a square matrix \mathbf{P} of order $m+1$ such that $\bar{\psi}(x_i) = \sum_{j=0}^m P_i^j y_j$ for $i = 0, 1, \dots, m$. In the proof of Theorem 5.1, we showed that the entries P_i^0 are all zero for $i = 1, \dots, m$ and that the entry P_0^0 is ± 1 when (i) $n \geq 2$ & $m \geq 2$ or (ii) $n = 1$ & $m \geq 3$. This implies that ψ induces a graded ring isomorphism from $H^*(S)$ to $H^*(S')$. \square

Theorem 5.4. *Let M be a quasitoric manifold, and let $P(E)$ be a projective bundle over a smooth compact toric surface S with fiber $\mathbb{C}P^n$. Assume that $n \geq 2$ or $b_2(S) \geq 2$. Then, M is equivalent to some projective bundle $P(E')$ over S' , where S' is a toric surface diffeomorphic to S .*

Before we prove the above theorem, we first investigate the classification of 4-dimensional quasitoric manifolds. By [21], a 4-dimensional quasitoric manifold is equivariantly diffeomorphic to an equivariant connected sum of several copies of $\mathbb{C}P^2$ and Hirzebruch surfaces. Since Hirzebruch surfaces are diffeomorphic to either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, any 4-dimensional quasitoric manifold is diffeomorphic to a connected sum of several copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Recall that the diffeomorphism classes of smooth 4-dimensional toric varieties are represented by $\mathbb{C}P^2$, the product of spheres $S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$, and the connected sum of $\mathbb{C}P^2$ with a finite number of copies of $\overline{\mathbb{C}P^2}$. Furthermore, the cohomology ring of each class is distinct.

Therefore, if a 4-dimensional quasitoric manifold N is diffeomorphic to a smooth compact toric surface, then N is equivalent to the toric manifold that is obtained from $\mathbb{C}P^2$ or one of the Hirzebruch surfaces by a succession of blow-ups.

Proof of Theorem 5.4. Set $m = b_2(S)$. If $m = 2$ and $n = 1$, $P(E)$ is a three-stage Bott manifold, and as mentioned previously, M is equivalent to some three-stage Bott manifold, which is a projective bundle.

Now, we assume that (i) $n \geq 2$ or (ii) $n = 1$ and $m \geq 3$. Then, by Theorem 5.1, M is equivalent to some equivariant bundle over a 4-dimensional quasitoric manifold N . By Lemma 5.3, $H^*(N)$ should be isomorphic to $H^*(S)$ as graded rings. Therefore, N and S are diffeomorphic to either $S^2 \times S^2$ or $\mathbb{C}P^2 \# (m-1)\overline{\mathbb{C}P^2}$. Thus, N is equivalent to some toric surface S' (it is not necessary that S' be equivariantly diffeomorphic to S). Hence, by Lemma 3.3, the characteristic matrix of M can be written as by

$$(5.7) \quad \Lambda = \left(\begin{array}{ccc|cc|c|ccc} 1 & \cdots & 0 & 0 & 0 & -1 & a_{11} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & -1 & a_{n,1} & \cdots & a_{n,m} \\ \hline 0 & \cdots & 0 & 1 & 0 & 0 & -\Lambda_1^1 & \cdots & -\Lambda_1^m \\ 0 & \cdots & 0 & 0 & 1 & 0 & -\Lambda_2^1 & \cdots & -\Lambda_2^m \end{array} \right),$$

where $\begin{pmatrix} 1 & 0 & -\Lambda_1^1 & \cdots & -\Lambda_1^m \\ 0 & 1 & -\Lambda_2^1 & \cdots & -\Lambda_2^m \end{pmatrix}$ is the characteristic matrix of S' . Note that by Proposition 3.4, there is a projective bundle over S' whose characteristic function is equal to (5.7). Since a quasitoric manifold is determined by its characteristic function up to equivalence, we conclude that M is equivalent to a projective bundle $P(E')$ over S' . \square

We close the section by noting that Theorem 1.1 immediately follows Theorem 5.1 and Theorem 5.4.

6. PROOFS OF THEOREMS 1.2 AND 1.3

In this paper, for a closed connected manifold M , we use $p(M)$ and $w(M)$ to denote the total Pontryagin class and the total Stiefel-Whitney class of M , respectively. In order to classify projective bundles over a smooth compact toric surface, we prepare a few lemmas that indicate the invariance of characteristic classes under cohomology ring isomorphisms.

Lemma 6.1. *If S and S' are 4-dimensional quasitoric manifolds such that $\phi: H^*(S) \rightarrow H^*(S')$ is a graded ring isomorphism, then ϕ preserves their Pontryagin classes, that is, $\phi(p(S)) = p(S')$.*

Proof. Note that the graded cohomology ring isomorphism ϕ induces the isomorphism $\varphi: H^2(S) \rightarrow H^2(S')$ that preserves the self-intersection form, and hence, the first Pontryagin class is preserved by φ . Therefore, one can see that $\phi(p(S)) = p(S')$. \square

Proposition 6.2. [9] *Let $E \rightarrow B$ and $E' \rightarrow B'$ be complex vector bundles over smooth manifolds B and B' with the same fiber dimension, respectively. Suppose that $\psi: H^*(P(E')) \rightarrow H^*(P(E))$ is an isomorphism such that $\psi(H^*(B')) = H^*(B)$ and $\psi(p(B')) = p(B)$; then, $\psi(p(P(E')) = p(P(E))$.*

Lemma 6.3. [9] *Let M be a connected closed manifold of dimension n . Suppose that $H^*(M)$ is generated by $H^r(M)$ for some r as a ring, and let M' be another connected manifold of dimension n such that $H^*(M'; \mathbb{Z}/2)$ is isomorphic to $H^*(M; \mathbb{Z}/2)$ as rings. Then, $\psi(w(M')) = w(M)$ for any ring isomorphism $\psi: H^*(M'; \mathbb{Z}/2) \rightarrow H^*(M; \mathbb{Z}/2)$.*

Theorem 6.4. *Let M and M' be projective bundles over smooth compact toric surfaces S and S' with the fiber space $\mathbb{C}P^1$, respectively. If $H^*(M) \cong H^*(M')$, then M and M' are diffeomorphic.*

Proof. If the orbit space of a smooth compact toric surface S is either Δ^2 or $\Delta^1 \times \Delta^1$, then M is equivalent to either a two-stage generalized Bott manifold or a three-stage Bott manifold. In these two cases, they are classified by their cohomology rings up to diffeomorphism in [9].

We only need to prove the case when M is neither a two-stage generalized Bott manifold nor a three-stage Bott manifold. That is, the orbit spaces of S and S' are $G(m+2)$ with $m \geq 3$. By Lemma 5.3, any graded ring isomorphism ψ from $H^*(M)$ to $H^*(M')$ induces an isomorphism between $H^*(S)$ and $H^*(S')$. Then, by Lemma 6.1, ψ satisfies $\psi(p(S)) = p(S')$. Hence, by Proposition 6.2, ψ satisfies $\psi(p(M)) = p(M')$. By Lemma 6.3, ψ also

preserves Stiefel-Whitney classes. Hence, the isomorphism ψ preserves their Stiefel-Whitney classes and Pontryagin classes. Therefore, the 6-dimensional toric manifolds M and M' are diffeomorphic by [20] and [23]. \square

Lemma 6.5. [22] *Let X be a finite CW-complex such that $H^{\text{odd}}(X) = 0$ and $H^*(X)$ has no torsion. Then, complex n -dimensional vector bundles over X with $2n \geq \dim X$ are isomorphic if and only if their total Chern classes are the same.*

Theorem 6.6. *Let $P(E)$ and $P(E')$ be projective bundles over smooth compact toric surfaces S and S' with the fiber space $\mathbb{C}P^n$, $n \geq 2$, respectively. If $H^*(P(E))$ is isomorphic to $H^*(P(E'))$, then $P(E)$ and $P(E')$ are diffeomorphic provided $b_2(S) \leq 10$.*

Proof. Let $m = b_2(S)$. If $m = 1$, i.e., the orbit spaces of S and S' are Δ^2 , $P(E)$ and $P(E')$ are two-stage generalized Bott manifolds. Thus, in this case, the statement is true by Theorem 6.1 in [9].

Now, we consider the case when the orbit spaces of S and S' are $G(m+2)$ with $m \geq 2$.

We first claim that there exist a line bundle L over S' and a diffeomorphism $g: S' \rightarrow S$ such that $g^*E \otimes L$ or its dual has the same total Chern class as E' . Let $E = \underline{\mathbb{C}} \oplus L_1 \oplus \cdots \oplus L_n$ with $c(L_i) = (1 + \sum_{j=1}^m a_{ij}x_j) \in H^*(S)$, and $E' = \underline{\mathbb{C}} \oplus L'_1 \oplus \cdots \oplus L'_n$ with $c(L'_i) = (1 + \sum_{j=1}^m a'_{ij}x_j) \in H^*(S')$. Then by (2.1), we have

$$H^*(P(E)) = H^*(S)[x_0] \left/ y_0 \prod_{i=1}^n \left(x_0 + \sum_{j=1}^m a_{ij}x_j \right) \right.,$$

and

$$H^*(P(E')) = H^*(S')[y_0] \left/ y_0 \prod_{i=1}^n \left(y_0 + \sum_{j=1}^m a'_{ij}y_j \right) \right..$$

From Lemma 5.3, if ψ is a graded ring isomorphism from $H^*(P(E))$ to $H^*(P(E'))$, then ψ induces a graded ring isomorphism $H^*(S) \rightarrow H^*(S')$. Note that since both S and S' are toric surfaces, according to Fischli and Yavin [15], both S and S' are diffeomorphic to either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# (m-1)\overline{\mathbb{C}P^2}$. Therefore, if $b_2(S) \leq 10$, any cohomology ring isomorphism $H^*(S) \rightarrow H^*(S')$ is induced by a diffeomorphism $S' \rightarrow S$ (see [6] and [24]). Let $g: S' \rightarrow S$ be the diffeomorphism induced from ψ .

Furthermore, we may assume that $\psi(x_0) = \epsilon y_0 + \alpha$, where $\epsilon = \pm 1$ and $\alpha \in H^2(S')$. Letting $u_i = \sum_{j=1}^n a_{ij}x_j$, $u'_i = \sum_{j=1}^n a'_{ij}y_j$ for $i = 1, \dots, n$, and $u_0 = u'_0 = 0$, we have $c(E) = \prod_{i=0}^n (1 + u_i)$ and $c(E') = \prod_{i=0}^n (1 + u'_i)$. Since $\psi(x_0) = \epsilon y_0 + \alpha$ and $\psi(x_i) \in H^*(S')$ for $i = 1, \dots, m$, we have

$$\psi\left(\prod_{i=0}^n (x_0 + u_i)\right) = \prod_{i=0}^n (\epsilon y_0 + \alpha + \psi(u_i)).$$

Since this element vanishes in $H^*(P(E'))$ and is a polynomial of degree $n+1$ in y_0 , we have an identity

$$\prod_{i=0}^n (\epsilon y_0 + \alpha + \psi(u_i)) = \epsilon^{n+1} \prod_{i=0}^n (y_0 + u'_i)$$

as polynomials in y_0 . Then, by plugging $y_0 = 1$, we obtain the identity

$$(6.1) \quad \prod_{i=0}^n (1 + \epsilon(\psi(u_i) + \alpha)) = \prod_{i=0}^n (1 + u'_i).$$

Note that $c(g^*E) = \prod_{i=0}^n (1 + \psi(u_i))$. Let L be the line bundle over S' whose first Chern class is $\alpha \in H^2(S')$. Then, (6.1) implies that either

$$c(g^*E \otimes L) = c(E') \text{ or } c((g^*(E) \otimes L)^*) = c(E').$$

This proves the claim.

By the above claim and Lemma 6.5, $g^*E \otimes L$ or its dual is isomorphic to E' . Since the induced bundle g^*E is isomorphic to E , and $P(g^*E)$ is isomorphic to both $P(g^*E \otimes L)$ and $P((g^*E \otimes L)^*)$ as bundles by Lemma 2.1, $P(E)$ is diffeomorphic to $P(E')$. \square

We obtain Theorem 1.2 by combining Theorem 6.4 and Theorem 6.6.

We remark that when S is $\mathbb{C}P^2 \# (m-1)\overline{\mathbb{C}P^2}$ with $m > 10$, by [17], there exists a self-isomorphism on $H^*(S)$ that cannot be induced from a self-diffeomorphism on S .

From Theorem 5.4, we obtain the following corollary by combining it with Theorem 6.4 and Theorem 6.6.

Corollary 6.7. *Let N and N' be quasitoric manifolds such that $H^*(N) \cong H^*(N') \cong H^*(P(E))$ for some projective bundle $P(E)$ over a smooth compact toric surface S . Then, N and N' are homeomorphic provided $n = 1$ or $b_2(S) \leq 10$.*

We can extend Theorem 5.1, Theorem 5.4, and Theorem 6.4 to the case when $P(E)$ is a projective bundle over a 4-dimensional quasitoric manifold, excluding a $\mathbb{C}P^1$ -bundle over $\mathbb{C}P^2 \# \mathbb{C}P^2$. This is because we never use the essential properties of smooth compact toric surfaces unless $P(E)$ is a generalized Bott manifold. Furthermore, Theorem 6.6 can be extended to the case when $P(E)$ is a projective bundle over a 4-dimensional quasitoric manifold S with $b_2(S) \leq 10$, since every cohomology ring isomorphism from $H^*(S)$ to itself is induced by a self-diffeomorphism on S by Wall (see page 137 in [24]). Hence, we obtain the following, which restates Theorem 1.3.

Theorem 6.8. *Let $P(E)$ be a projective bundle over a 4-dimensional quasitoric manifold S with fiber $\mathbb{C}P^n$. Assume that $n = 1$ or $b_2(S) \leq 10$. Let M be a quasitoric manifold whose cohomology ring is isomorphic to $H^*(P(E))$. Then, M is homeomorphic to $P(E)$ unless $P(E)$ is a $\mathbb{C}P^1$ -bundle over $\mathbb{C}P^2 \# \mathbb{C}P^2$.*

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